

A Linear-Time Algorithm for Maximum-Cardinality Matching on Cocomparability Graphs

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Abstract

Finding maximum-cardinality matchings in undirected graphs is arguably one of the most central graph problems. For general m -edge and n -vertex graphs, it is well-known to be solvable in $O(m\sqrt{n})$ time. We develop the first linear-time algorithm to find maximum-cardinality matchings on *cocomparability* graphs, a prominent subclass of perfect graphs that contains interval graphs as well as permutation graphs. Our algorithm is based on the recently discovered *Lexicographic Depth First Search (LDFS)*.

1 Introduction

The problem MATCHING (or MAXIMUM CARDINALITY MATCHING) is, given an undirected graph, to compute a maximum-cardinality set of disjoint edges. MATCHING is arguably among the most fundamental graph-algorithmic primitives that can be computed in polynomial time. More specifically, the fastest known algorithm for computing a maximum-cardinality matching on an n -vertex and m -edge graph runs in $O(m\sqrt{n})$ time [26]. No faster algorithm is known even when the given graph is bipartite [16]. Improving this running time, either on general graphs or on bipartite graphs, resisted decades of research. In terms of approximation, it is known that the $O(m\sqrt{n})$ algorithm of Micali and Vazirani [26] implies an $(1 - \epsilon)$ -approximation running in $O(m\epsilon^{-1})$ time [9]. For the weighted case, recently Duan and Pettie [9] provided a linear-time algorithm that computes an $(1 - \epsilon)$ -approximate maximum-weight matching (the constant running time factor depending on ϵ is $\epsilon^{-1} \log(\epsilon^{-1})$). In this work we take a route different to approximation and identify a large graph class, namely cocomparability graphs, on which we show that an optimal solution can be computed in linear time.

To identify more efficiently solvable special cases for finding maximum matchings has quite some history. For instance, Yuster [35] developed an algorithm with running time $O(rn^2 \log n)$, where r denotes the difference between maximum and minimum vertex degree of the input graph. Moreover, there are (quasi)*linear-time* algorithms for computing maximum matchings in several special classes of graphs, including interval graphs [20], convex bipartite graphs [33], strongly chordal graphs [7], and chordal bipartite graphs [2]. We refer to Table 1 for a more thorough overview, also including results with superlinear running times.

A graph G is a *cocomparability* graph if its complement \overline{G} admits a transitive orientation of its edges (that is, of the non-edges of G). These graphs (as well as their complements, i.e. comparability graphs) arise naturally in several real-world applications as they correspond one-to-one to partially ordered sets; moreover they have been subject of intensive theoretical research [3,

^{*}Supported by a postdoc fellowship of the German Academic Exchange Service (DAAD) while at Durham University.

Table 1: Fastest algorithms for MATCHING on special graph classes; $\omega < 2.373$ is the matrix multiplication exponent, that is, two $n \times n$ matrices can be multiplied in $O(n^\omega)$ time.

graph class	running time
general	$O(m\sqrt{n})$ [26], $O(m\sqrt{n} \log(n^2/m)/\log(n))$ [14], $O(n^\omega)$ (rand.) [28]
bipartite	$O(m\sqrt{n})$ [16], $O(n^\omega)$ (rand.) [28, 31], $O(m^{1.43})$ [22]
interval	$O(n \log n)$ (given an interval representation) [20, 27]
circular arcs	$O(n \log n)$ [20]
co-interval	$O(n \log n + m)$ [13]
convex bipartite	$O(n)$ [33]
planar	$O(n^{\omega/2})$ (rand.) [29]
strongly chordal	$O(n + m)$ (given the strong perfect elimination order) [7]
chordal bipartite	$O(n + m)$ [2]
regular	$O(n^2 \log n)$ [35]
cographs	$O(n)$ (given a co-tree) [34]
co-comparability	$O(n + m)$ (Theorem 2.5 in Section 2)

5, 6, 8, 10, 11, 17–19, 24]. On the one hand, cocomparability graphs naturally generalize well-studied graph classes such as interval graphs and permutation graphs [1, 15]. On the other hand, cocomparability graphs form an almost maximal subclass of perfect graphs [1]¹. Since perfect graphs (as well as comparability graphs) properly contain bipartite graphs, it seems out of reach to obtain an algorithm for MATCHING with linear running time on perfect graphs. Therefore, designing a linear-time algorithm for cocomparability graphs provides a sharp boundary between $O(n + m)$ -time algorithms and $O(m\sqrt{n})$ -time algorithms for MATCHING.

Our contribution In this paper we present the first linear-time algorithm for MATCHING on *cocomparability* graphs. It is a simple greedy algorithm, referred to as *Rightmost Matching (RMM)*, running on a specific vertex ordering σ . This ordering is obtained by using (as a preprocessing step) the recently discovered *Lexicographic Depth First Search (LDFS)* algorithm [4]. Interestingly, although the proof of correctness of RMM on cocomparability graphs is technically involved, it turns out that RMM computes in a *trivial* way a maximum matching on interval graphs, when applied on the standard interval graph vertex ordering². Recall that the class of interval graphs is a strict subset of the class of cocomparability graphs. So far a similar phenomenon of extending an interval graph algorithm to cocomparability graphs by using an LDFS preprocessing step has only been observed for the LONGEST PATH problem [24] and for the MINIMUM PATH COVER problem [5]. Our results for the RMM algorithm, in addition to the previous results [5, 24], provide evidence that cocomparability graphs present an “interval graph structure” when they are considered with an LDFS preprocessing step. This insight might lead to new and more efficient combinatorial algorithms.

Preliminaries We use standard notation from graph theory. In particular, all paths we consider are simple paths. A *matching* in a graph is a set of pairwise disjoint edges. Let $G = (V, E)$ be a graph and let $M \subseteq E$ be a matching in G . A vertex $v \in V$ is called *matched* with respect to M if there is an edge in M containing v , otherwise v is called *free* with respect to M . If the matching M is clear from the context, then we omit “with respect to M ”. An *alternating path* with respect to M is a path in G such that every second edge of the path belongs to M . An *augmenting path* is an alternating path whose endpoints are free. It is well-known that a matching M is maximum if and only if there is no augmenting path for it [21].

¹For an updated overview of the relation between graph classes see <http://www.graphclasses.org/>.

²This is the vertex ordering that results by sorting the intervals according to their left endpoints. The RMM algorithm for interval graphs was discovered by Moitra and Johnson [27].

A graph $G = (V, E)$ is an *interval graph* if we can assign to each vertex of G a closed interval on the real line such that two vertices are adjacent in G if and only if the corresponding two intervals intersect. A *comparability graph* is a graph whose edges can be transitively oriented, i. e. if $u \rightarrow v$ and $v \rightarrow w$ then $u \rightarrow w$. A *cocomparability graph* G is a graph whose complement \overline{G} is a comparability graph. The class of interval graphs is strictly included in the class of cocomparability graphs [1]. Intuitively, we can transitively orient the “non-edges” of an interval graph using the ordering of non-intersecting intervals from left to right, as follows. Consider three intervals I_a, I_b, I_c in an interval representation of an interval graph. If I_a lies completely to the left of I_b , and I_b lies completely to the left of I_c , then also I_a lies completely to the left of I_c .

2 A linear-time algorithm for cocomparability graphs

To begin with, we first present in [Section 2.1](#) a simple greedy linear-time algorithm for computing a maximum matching M on interval graphs and then we prove in [Section 2.2](#) that the same algorithm actually works also for cocomparability graphs.

2.1 The simple algorithm for interval graphs

Given an interval graph G with n vertices and m edges, we first compute in $O(n + m)$ time an interval representation of G and, at the same time, we also sort the intervals according to their left endpoint [30]. The algorithm works as follows (cf. [20, 27]):

1. Initialize $M = \emptyset$ and label all vertices as “unvisited”.
2. Pick the unvisited vertex (interval) x which has the rightmost left endpoint among all currently unvisited vertices in G . Then, label x as “visited”.
3. If x has at least one unvisited neighbor in G , then pick the unvisited neighbor y of x which has the rightmost left endpoint among all unvisited neighbors of x . Then label y as “visited” and add the edge $\{x, y\}$ to the computed matching M .
4. If there is still an unvisited vertex in G , then go to [Step 2](#).
5. Return the maximum matching M .

We call the above algorithm *Rightmost-Matching (RMM)*. It can be executed in $O(n + m)$ time; with a simple exchange argument we can show that the matching M returned by RMM is indeed maximum in G . This algorithm implicitly uses the following vertex ordering that characterizes interval graphs. This ordering corresponds to sorting the intervals according to their left endpoints and can be computed in $O(n + m)$ time from G [30].

Lemma 2.1 ([30]). *$G = (V, E)$ is an interval graph if and only if there exists a vertex ordering σ of G (called an I-ordering) such that, for all $x <_\sigma y <_\sigma z$, if $\{x, z\} \in E$, then also $\{x, y\} \in E$.*

2.2 The algorithm for cocomparability graphs

As our central result of this paper, we prove below that the simple RMM algorithm presented in [Section 2.1](#) can actually compute a maximum matching in the much greater class of cocomparability graphs, once it is applied to a specific vertex ordering σ ; see [Algorithm 1](#) for a description of RMM when the input is a cocomparability graph. This ordering σ is obtained by applying (as a preprocessing step) the recently discovered *Lexicographic Depth First Search (LDFS)* algorithm [4] to a vertex ordering π that characterizes cocomparability graphs (see [Definition 1](#) and [Lemma 2.2](#)). LDFS is a variation of the well-known Depth First Search (DFS) algorithm; LDFS appropriately assigns labels to the vertices and uses the lexicographic order over these labels as a tie-breaking rule.

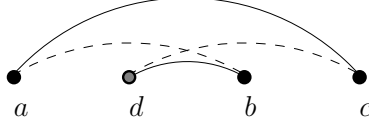


Figure 1: A good triple (a, b, c) and its d -vertex, in the vertex ordering $\sigma = (a, d, b, c)$. The edges $\{a, c\}$ and $\{d, b\}$ are indicated with solid lines and the non-edges $\{a, b\}$ and $\{d, c\}$ with dashed lines.

Before we proceed with our algorithm and its analysis, we first present in [Definition 1](#) the *umbrella-free* ordering that characterizes cocomparability graphs. This ordering is a direct generalization of the I-ordering for interval graphs, see [Lemma 2.1](#). It is worth noting here that, although there exists a linear-time algorithm to *compute* an umbrella-free ordering π of a given cocomparability graph [23], the fastest known algorithm to *verify* that a given vertex ordering is indeed umbrella-free needs the same time as boolean matrix multiplication (Spinrad [32] discusses this issue).

Definition 1 ([18]). Let $G = (V, E)$ be a graph. An ordering π of the vertices V is an *umbrella-free* ordering (or a *CO-ordering*) if for all $x <_{\pi} y <_{\pi} z$ it holds that if $\{x, z\} \in E$, then $\{x, y\} \in E$ or $\{y, z\} \in E$ (or both).

Lemma 2.2 ([18]). A graph $G = (V, E)$ is a cocomparability graph if and only if there exists an *umbrella-free* ordering π of V .

As a first step for our algorithm we compute from the given umbrella-free ordering π a new vertex ordering σ by applying a preprocessing step to π which is based on the *LDFS* algorithm [4]. In short, the LDFS algorithm is a particular kind of a DFS execution which sequentially assigns certain labels to the visited vertices. Then, whenever there is a tie among several vertices to visit next, the LDFS algorithm always chooses one vertex with the currently *lexicographically largest* label. We present in [Definition 3](#) below the notion of an *LDFS-ordering*. It has been proven that a vertex ordering σ is an LDFS-ordering if and only if σ can be returned by an application of the LDFS algorithm [4].

The notion of an LDFS ordering σ (see [Definition 3](#)) is based on the notions of a *good triple* and a *bad triple* (see [Definition 2](#)). Intuitively, both a good and a bad triple with respect to a vertex ordering σ are triples of vertices $a, b, c \in V$ such that the restriction of σ is *not* an I-ordering (see [Lemma 2.1](#)). Such a triple a, b, c of vertices is *good* if there exists a fourth vertex d with special properties in the ordering σ ; otherwise the triple is *bad*. Then, σ is an LDFS ordering if it has no bad triples. For the purposes of our algorithm for MATCHING, whenever σ is simultaneously an umbrella-free ordering and an LDFS ordering, it turns out that σ behaves with the RMM algorithm in exactly the same way as an I-ordering, thus solving MATCHING in linear time on cocomparability graphs.

Definition 2 ([4]). Let $G = (V, E)$ be a graph and σ an ordering of V . Let $a, b, c \in V$ be three vertices such that $a <_{\sigma} b <_{\sigma} c$, $\{a, c\} \in E$, and $\{a, b\} \notin E$. If there exists a vertex d such that $a <_{\sigma} d <_{\sigma} b$, $\{d, b\} \in E$, $\{d, c\} \notin E$, then (a, b, c) is a *good triple*, otherwise it is a *bad triple*. Furthermore, for a good triple (a, b, c) , this vertex d is called its *d-vertex*.

Definition 3 ([4]). Let $G = (V, E)$ be a graph. An ordering σ of V is an *LDFS ordering* if σ has no bad triple.

An example of a good triple (a, b, c) and the corresponding d -vertex is depicted in [Figure 1](#). Note that the input to the standard LDFS algorithm is a graph G and a starting vertex v . As an additional tie-breaking rule (usually referred to as a “+–rule”), LDFS can take as input also a given vertex ordering π . In this case, LDFS⁺ visits at every step the *rightmost* unvisited vertex in π among all vertices that have the currently lexicographically largest label. The resulting ordering σ

is then denoted $\sigma = \text{LDFS}^+(G, \pi)$; see [4, 5, 24] for a more detailed discussion. Given an umbrella-free ordering π of a cocomparability graph G with n vertices and m edges, $\text{LDFS}^+(G, \pi)$ can be computed in $O(n + m)$ time [17]. In the preprocessing step for our algorithm we compute the ordering $\sigma = \text{LDFS}^+(G, \pi)$ where π is the given umbrella-free ordering of G . Then, this LDFS vertex ordering σ is also umbrella-free [5].

Once we have computed in linear time the LDFS umbrella-free ordering $\sigma = \text{LDFS}^+(G, \pi)$, we apply our simple linear-time algorithm *Rightmost-Matching (RMM)*, see Algorithm 1, to compute a new vertex ordering $\hat{\sigma}$ and a maximum matching M of G . RMM is a simple “greedy” algorithm which operates as follows. At every step it visits the rightmost unvisited vertex x in σ and it labels x as visited. Then, if x does not have any unvisited neighbor, then RMM proceeds at the next step by visiting again the currently unvisited vertex in σ ; note that this vertex is now different from x , as x has been already labeled as visited. Otherwise, if x has at least one unvisited neighbor, then RMM visits after x its rightmost unvisited neighbor y in the ordering σ and it also adds the edge $\{x, y\}$ to the computed matching M .

Algorithm 1 $\text{RMM}(G, \sigma)$.

Input: A cocomparability graph G with an LDFS umbrella-free ordering σ of G .

Output: A vertex ordering $\hat{\sigma}$ of G and a maximum matching of G .

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1: Label all vertices “unvisited”;  $i \leftarrow 0$ ;  $M \leftarrow \emptyset$ 
2: while there are unvisited vertices do
3:   Pick the rightmost unvisited vertex  $x$  in  $\sigma$  and label  $x$  as “visited”
4:    $i \leftarrow i + 1$ ;  $\hat{\sigma}(i) \leftarrow x$                                 {add vertex  $x$  to the ordering  $\hat{\sigma}$ }
5:   if  $x$  has at least one unvisited neighbor then
6:     Pick the rightmost unvisited neighbor  $y$  of  $x$  and label  $y$  as “visited”
7:      $i \leftarrow i + 1$ ;  $\hat{\sigma}(i) \leftarrow y$                             {add vertex  $y$  to the ordering  $\hat{\sigma}$ }
8:      $M \leftarrow M \cup \{\{x, y\}\}$                                 {match  $x$  and  $y$ }
9: return the ordering  $\hat{\sigma}$  and the matching  $M$ 

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Remark 1. Since any I-ordering of an interval graph is also an LDFS umbrella-free ordering (see Lemma 2.1 and Definitions 1 to 3), note that Algorithm 1 also works with an interval graph G and an I-ordering σ of G as input. In this case, $\text{RMM}(G, \sigma)$ is actually exactly the same RMM algorithm as we sketched in Section 2.1 for interval graphs.

In the remainder of this section, we show that the matching M returned by $\text{RMM}(G, \sigma)$ is indeed a maximum matching of G . The proof is by contradiction and uses an appropriate *potential function* f that is defined over all matchings of G , see Definition 4.

Definition 4 (potential function). Let $G = (V, E)$ be a cocomparability graph and σ be an LDFS umbrella-free ordering of $V = \{v_1, \dots, v_n\}$ with $v_1 <_\sigma \dots <_\sigma v_n$. Let M be a matching of G . We define the potential function $f(M) := \sum_{i=1}^n g(v_i)$, where for each $v_i \in V$:

$$g(v_i) := \begin{cases} 0, & \text{if } \{v_i, v_j\} \in M \text{ and } i < j, \\ (i - j) \cdot (n + 1)^i, & \text{if } \{v_i, v_j\} \in M \text{ and } j < i, \\ i \cdot (n + 1)^i, & \text{if } v_i \text{ is not matched within } M. \end{cases}$$

Note by Definition 4 that, for the empty matching, we have $f(\emptyset) = \sum_{i=1}^n i \cdot (n + 1)^i$. Then, as we add an edge $\{\{v_i, v_j\}\}$ to the current matching M , where $j < i$, we have that $f(M \cup \{\{v_i, v_j\}\}) = f(M) - j \cdot (n + 1)^i$. The next observation illustrates the connection between the RMM algorithm and this potential function.

Observation 2.3. *The matching M returned by $\text{RMM}(G, \sigma)$ minimizes the function $f(M)$.*

Proof. Let M' be a matching such that $f(M')$ is minimum. Consider the rightmost vertex v_n in the ordering σ and let v_i be the rightmost neighbor of v_n in σ . Assume that $\{v_i, v_n\} \notin M'$. Then

let M'' be the matching obtained by removing from M' the edges that v_i and v_n are matched to, and by adding to it the edge $\{v_i, v_n\}$. Then it is easy to see that $f(M'') < f(M')$ which is a contradiction. Thus $\{v_i, v_n\} \in M'$. We can now recursively apply the same argument in the induced subgraph $G[V \setminus \{v_i\} \setminus \{v_n\}]$, which eventually implies that M' is the matching returned by $\text{RMM}(G, \sigma)$. \square

Before we prove our main result in [Theorem 2.5](#), we need to prove a crucial technical lemma ([Lemma 2.4](#)). On a high level, our strategy for the proofs is as follows: We consider a maximum matching M minimizing f and a matching M' produced by [Algorithm 1](#). If $M = M'$, then we are done. Otherwise, we take the “rightmost” difference between M and M' , that is the rightmost vertex v in M that is not matched in the same way in M' (or v is free in exactly one of the two matchings). Then we show that matching v in M as in M' leads to another maximum matching M'' such that $f(M'') < f(M)$. For this part we need a case distinction where in two cases we need to exclude the special scenario described in [Lemma 2.4](#). The existence of M'' is a contradiction to our choice of M and shows that $M = M'$.

In the next definition we introduce for every vertex v the induced subgraph $G_\sigma(v)$ with respect to the ordering σ , which is fundamental for the statement and the proof of [Lemma 2.4](#).

Definition 5. Let $G = (V, E)$ be a cocomparability graph and let $\sigma = (v_1, v_2, \dots, v_n)$ be an LDFS umbrella-free vertex ordering of G . Then, for every $v_i \in V$, the graph $G_\sigma(v_i)$ is the induced subgraph of G on the vertices $\{v_1, v_2, \dots, v_i\}$.

Lemma 2.4. Let $G = (V, E)$ be a cocomparability graph and σ be an LDFS umbrella-free ordering of V . Let M be a maximum matching of G such that $f(M)$ is minimum among all maximum matchings. Then, there is no quadruple (a, b, c, x) of vertices in G satisfying all of the following conditions:

1. $a <_\sigma b <_\sigma c \leq_\sigma x$,
2. $\{a, c\}, \{b, c\} \in E$ and $\{a, b\} \notin E$,
3. $\{a, c\} \in M$,
4. there is no odd-length alternating path from a to b within $G_\sigma(x)$,
5. there is no odd-length alternating path from a to any free vertex v within $G_\sigma(x)$, and
6. there is no odd-length alternating path from b to any free vertex v within $G_\sigma(x)$.

Proof. Let G , σ , and M as described in the statement of the lemma. The proof is by contradiction. Towards a contradiction let (a, b, c, x) be a quadruple of vertices satisfying all six conditions of the lemma. Fix now vertex x . Among all such quadruples with fixed x , let (a, b, c, x) be such that a is leftmost in σ , that is, for any other such quadruple (a', b', c', x) we have $a \leq_\sigma a'$. Since σ is an LDFS ordering, it follows from [Conditions 1](#) and [2](#) and [Definitions 2](#) and [3](#) that there is a vertex d such that $a <_\sigma d <_\sigma b$, $\{d, b\} \in E$, and $\{d, c\} \notin E$. Since $\{d, c\} \notin E$ and σ is umbrella-free, it follows that $\{a, d\} \in E$. Observe that d is matched in M as otherwise [Conditions 5](#) and [6](#) would be violated. Thus, there is a vertex $e \in V$ with $\{e, d\} \in M$. Now we distinguish three cases with respect to the position of e in the ordering σ .

Case 1: $c <_\sigma e$. In this case we have that $a <_\sigma c <_\sigma e$, $\{d, e\} \in E$, and $\{d, c\} \notin E$. Thus, since σ is umbrella-free, it follows that $\{c, e\} \in E$. However, in this case for the matching $M' = (M \setminus \{\{a, c\}, \{d, e\}\}) \cup \{\{e, c\}, \{a, d\}\}$ we have that $f(M') < f(M)$, a contradiction.

Case 2: $a <_\sigma e <_\sigma c$. If $\{a, e\} \in E$, then there exists the length-three alternating path (a, e, d, b) from a to b within $G_\sigma(x)$, which is a contradiction to [Condition 4](#). Thus, $\{a, e\} \notin E$. Furthermore, $\{c, e\} \in E$, since σ is umbrella-free and $\{a, c\} \in E$. Hence, for the matching $M' = (M \setminus \{\{a, c\}, \{d, e\}\}) \cup \{\{e, c\}, \{a, d\}\}$ we have that $f(M') < f(M)$, a contradiction.

Case 3: $e <_\sigma a$. In this case it follows similarly to Case 2 that $\{a, e\} \notin E$ (proof by contradiction due to [Condition 4](#)). Furthermore observe that $\{e, d\}, \{a, d\} \in E$ and $\{e, d\} \in M$.

Thus the triple (e, a, d) satisfies **Conditions 1 to 3**. Furthermore, if there exists an odd-length alternating path from e to a within $G_\sigma(x)$, then this alternating path can be extended through d to an odd-length alternating path from e to b within $G_\sigma(x)$, which is a contradiction to **Condition 4**. Hence there is no odd-length alternating path from e to a within $G_\sigma(x)$. Similarly, odd-length alternating paths from e (resp. from a) to a free vertex v within $G_\sigma(x)$ are excluded as well due to **Condition 5** (resp. due to **Condition 6**). Thus the quadruple (e, a, d, x) satisfies the six conditions of the lemma and it holds that $e <_\sigma a$, a contradiction to the choice of the initial quadruple (a, b, c, x) . \square

We are now ready to prove our central result.

Theorem 2.5. *For any n -vertex and m -edge cocomparability graph G , **Algorithm 1** returns a maximum matching M of G in $O(n + m)$ time.*

Proof. Let $G = (V, E)$ be a cocomparability graph, and let σ be an umbrella-free LDFS ordering of G . First we prove that **Algorithm 1** runs in $O(n + m)$ time. During the execution of the algorithm we maintain the *unvisited vertices* in an array A (initially of size n), according to their position in σ . Furthermore, we maintain for each vertex u its *unvisited neighbors* in an array N_u (initially of size $\deg(u)$), again according to their position in σ . Once we have computed the ordering σ , the construction of array A can be done in $O(n)$ time. The construction of all arrays N_u , where $u \in V$, can be done in $O(n + m)$ time as follows. We initialize $N_u = \emptyset$ for every $u \in V$. Then we iterate for each vertex $u \in V$ in array A from left to right. For every such vertex u we scan (in an arbitrary order) through its neighborhood $N(u)$, and for each $v \in N(u)$ we append vertex u in the array N_v .

Line 1 can be clearly executed in $O(n)$ time. The rightmost unvisited vertex x in **Line 3** can be found in $O(1)$ time as the rightmost vertex in the array A . Once x is detected in **Line 3**, x is removed from A also in $O(1)$ time. Furthermore x is removed from all arrays N_u , where $\{x, u\} \in E$, in $O(\deg(x))$ time. Moreover, **Line 4** can be clearly executed in $O(1)$ time. The if-condition of **Line 5** can be checked in $O(1)$ time by just checking whether the array N_x is empty. Similarly to **Line 3**, **Line 6** can be executed in $O(\deg(y))$ time. Furthermore, each of **Lines 7 to 9** can be clearly executed in $O(1)$ time. Summarizing, the total running time of **Algorithm 1** is $O(n + \sum_{u \in V} \deg(u)) = O(n + m)$ time.

For the correctness part, the proof is done by contradiction. Let M be the matching returned by $\text{RMM}(G, \sigma)$. Assume towards a contradiction that M is not a maximum matching. For the rest of the proof, let M' denote a maximum matching that minimizes $f(M')$ among all maximum matchings of G . Let x be the rightmost vertex in σ on which M' differs from M . Then x is matched in at least one of the two matchings M and M' . Now we distinguish three cases with respect to the vertex that is matched with x in M and M' .

Case 1: x is matched in M' to some $y \in V$ but is free in M . Then M and M' also differ at vertex y . Thus $y <_\sigma x$, since x is the rightmost vertex in which M and M' differ. Consider the iteration t of **Algorithm 1** during which the algorithm visits x . Suppose that vertex y is matched in M with a vertex z at an earlier iteration $t' < t$. Then M differs from M' also at vertex z . If $z <_\sigma x$, then **Algorithm 1** visits x at an earlier iteration than z , which is a contradiction to the assumption on z . Hence $x <_\sigma z$. This is a contradiction to the assumption that x is the rightmost vertex in σ in which M differs from M' .

Case 2: x is matched in M to some vertex $y \in V$ but is free in M' . If y is free in M' , then the matching $M' \cup \{\{x, y\}\}$ is greater than M' , which is a contradiction to the maximality assumption on M' . Therefore y is matched in M' to some vertex $z \in V$. Note that M and M' differ also on y and z . Thus, it follows by the choice of x that $y <_\sigma x$ and $z <_\sigma x$. Consider now the matching $M'' := (M' \setminus \{\{y, z\}\}) \cup \{\{x, y\}\}$, which is maximum since $|M''| = |M'|$. However $f(M'') < f(M')$, which is a contradiction to the assumption on the minimality of $f(M')$.

Case 3: $\{x, y\} \in M$ and $\{x, z\} \in M'$ with $z \neq y$. Then M and M' differ also on y and z . Thus, it follows by the choice of x that $y <_\sigma x$ and $z <_\sigma x$. Consider the iteration t of **Algorithm 1** during which the algorithm visits x . Suppose that vertex z is matched in M with a vertex p at an earlier iteration $t' < t$. Then M differs from M' also at vertex p . If $p <_\sigma x$, then **Algorithm 1**

visits x at an earlier iteration than p , which is a contradiction to the assumption on p . If $x <_\sigma p$, then we have again a contradiction to the assumption that x is the rightmost vertex in σ in which M differs from M' . Thus z is unmatched in M at the iteration t of [Algorithm 1](#) during which the algorithm visits x . Furthermore, z is also unvisited at iteration t since $z <_\sigma x$. Now, if $y <_\sigma z$, then [Algorithm 1](#) would not match x to y at the execution of [Line 6](#), which is a contradiction. Hence $z <_\sigma y$.

Suppose that y is free in M' . Then $M'' := (M' \setminus \{\{x, z\}\}) \cup \{\{x, y\}\}$ is another maximum matching, for which $f(M'') < f(M')$, a contradiction to the assumption on M' . Hence, y is matched in M' to some vertex $w \in V$. If $\{w, z\} \in E$, then the matching $M'' := (M' \setminus \{\{x, z\}, \{w, y\}\}) \cup \{\{x, y\}, \{z, w\}\}$ is another maximum matching, for which $f(M'') < f(M')$, which is a contradiction to the choice of M' . Hence $\{z, w\} \notin E$.

Suppose that within $G_\sigma(x)$ ([Definition 5](#)) there exists an odd-length alternating path P_0 with respect to M' from w to z . Then, swapping all edges on the path P_0 , removing $\{x, z\}$ and $\{w, y\}$ from M' , and adding $\{x, y\}$ yields another maximum matching M'' . Recall that x is the rightmost vertex in which M and M' differ. Thus, since the alternating path P_0 belongs to the induced subgraph $G_\sigma(x)$, it follows that $f(M'') < f(M')$, a contradiction to the choice of M' . Thus, within $G_\sigma(x)$ there exists no odd-length alternating path with respect to M' from w to z .

Similarly, suppose that within $G_\sigma(x)$ there exists an odd-length alternating path P_1 with respect to M' from w (resp. from z) to a free vertex v . Then, swapping all the edges on the path P_1 , removing $\{x, z\}$ and $\{w, y\}$ from M' and adding $\{x, y\}$ yields another maximum matching M'' such that $f(M'') < f(M')$, which is again a contradiction to the choice of M' . Thus there exists within $G_\sigma(x)$ no odd-length alternating path with respect to M' from w (resp. from z) to a free vertex v .

Now suppose that $w <_\sigma z$. That is, $w <_\sigma z <_\sigma y$, where $\{w, y\} \in E$ and $\{z, w\} \notin E$. Hence, $\{z, y\} \in E$ since σ is umbrella-free. Thus, since $\{w, y\} \in M'$, it follows that the quadruple (w, z, y, x) satisfies all six conditions in the statement of [Lemma 2.4](#). This is a contradiction to [Lemma 2.4](#), since M' is assumed to be a maximum matching of G such that $f(M')$ is minimum among all maximum matchings.

Finally suppose that $z <_\sigma w$. Recall that M differs from M' in w , since $\{y, w\} \in M'$ and $\{x, y\} \in M$. Thus, since x is the rightmost vertex in σ in which M differs from M' , it follows that $w <_\sigma x$. That is, $z <_\sigma w <_\sigma x$, where $\{x, z\} \in E$ and $\{z, w\} \notin E$. Hence, $\{w, x\} \in E$ since σ is umbrella-free. Thus, since $\{x, z\} \in M'$, it follows that the quadruple (z, w, x, x) satisfies all six conditions in the statement of [Lemma 2.4](#). This is again a contradiction to [Lemma 2.4](#), since M' is assumed to be a maximum matching of G such that $f(M')$ is minimum among all maximum matchings.

Summarizing, the matching M returned by [Algorithm 1](#) is a maximum matching. \square

3 Conclusion

We presented a thorough mathematical analysis of an efficient and easy-to-implement linear-time algorithm for computing maximum matchings on cocomparability graphs. This provides a new contribution to a long list of polynomial-time algorithms for problems on cocomparability graphs. Notably, most of this previous work showed polynomial-time (typically far from linear) algorithms for on general graphs NP-hard problems, while we improved a problem solvable in polynomial time on general graphs to linear time on cocomparability graphs.

Apart from being of interest on its own, our result might also be useful in a more general approach towards deriving faster algorithms for computing maximum matchings in relevant special cases. The fundamental idea behind this, as described in detail in companion work [\[25\]](#), is as follows. First, observe that once given any matching that has k edges less than an optimal one, then using k iterated augmenting path computations (each taking linear time [\[12\]](#)) one can improve it to a maximum matching. If we now knew that a given input graph G is k vertex deletions away from a cocomparability graph, then for constant k we could get a linear-time algorithm for MATCHING as follows: First, delete the k vertices from G , then apply our linear-time algorithm,

and then apply (as described above) at most k iterations of augmenting path computations again with respect to the original graph G , starting with the maximum matching for the cocomparability graph. A drawback of this approach is that we do not know how to compute a constant-factor approximation (which would be good enough) for the mentioned vertex deletion set of size k . Hence, we consider it as an important challenge for future work to give a linear-time (constant-factor approximation) algorithm for computing a “minimum-vertex-deletion-to-cocomparability” set. For now, we only can state the following result:

Corollary 3.1. *MATCHING can be solved in $O(k \cdot (n + m))$ time when given a vertex set subset X of size k such that deleting X yields a cocomparability graph.*

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